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DYNAMIC PLANNING AND TWO- OR MULTISTAGE
PROGRAMMING UNDER RISK

by

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ABSTRACT

It is shown that the model of two- or multistage programming under risk results from the application of the principle of dynamic planning* in special structured dynamic decision situations under risk. By using this property, some characteristics of the solution set of dynamic planning problems will be derived.

*This principle is well known and discussed as "flexible Planung" in the German management literature in recent times.

1. Decision Situation

Present and future actions of a firm are usually interdependent. Actions (decisions) made in a present period are able to influence the set of feasible decisions in the following periods as well as the value of the objective function for the entire number of planning periods. If, for example, a firm wants to realize a proposed investment it has to decide how to use it in future periods to be able to make a decision at the present time of planning. So the decision situation which has to be considered in this paper is defined by several planning periods and decision instants. If the future development of the nature (the relevant data for the decision like cash flows) is certain, all future actions or decisions can be determined at the present time of planning. However, if the decision situation is defined by risk (the development of the nature in future periods is a random variable with known distribution function) it is no use to determine uniquely future actions (decisions) at the present time. The set of feasible decisions (the decision space) in future periods depends on the outcome of the random variable nature. Future decisions, uniquely determined at the present time of planning, are neither necessary feasible, nor optimal. Such dynamic decision situations under risk are defined by (Hax/Laux, 1972, p. 319).

Def. 1: a) Decisions have to be made at $H \geq 2$ succeeding points of time.

b) The nature at every point of time h ($h=1, 2, \dots, H$) is a random variable (at $h=1$ with only one outcome).

c) The set of feasible decisions (actions) - the decision space - at $\hat{h} \in \{2, 3, \dots, H\}$ depends on the decisions which have been made at $h=1, 2, \dots, \hat{h}-1$ and the

outcomes of the random variable at $h=1, 2, \dots, \hat{n}$.

If the nature is a discrete random variable, the decision situation, given by Def. 1 can easily be illustrated by using a state tree. For $H:=2$ we obtain for instance figure 1.

Figure 1

$h:=1$

decision vector: y^1

vector of the obj. fct.: s_1

nature: $\{s_1, M^1, b^1\}$

decision space: $T_1 = \{y^1 | M^1 y^1 = b^1; y^1 \geq 0\}$

transition probabilities

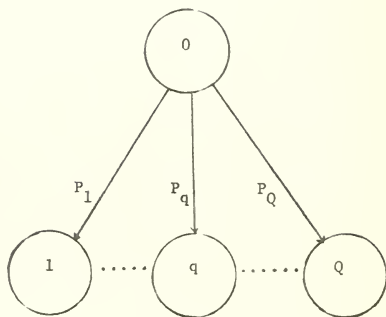
$h:=2$

decision vector: y^{2q}

vector of the obj. fct.: s_{2q}

nature: $\{s_{2q}, M^{2q}, b^{2q}, A^{2q}\}$

decision space: $T_{2q} = \{y^{2q} | M^{2q} y^{2q} = b^{2q} - A^{2q} y^1; y^{2q} \geq 0\}$



The nature at $h:=1$ has a degenerate distribution function. The only outcome is defined by the matrix M^1 and the vectors s_1, b^1 with the probability one.

At $h:=2$ the nature is a discrete random variable with the Q outcomes $\{s_{2q}, M^{2q}, b^{2q}, A^{2q}\}$ and the probabilities P_q . M^{2q}, A^{2q} mean matrices and b^{2q}, s_{2q} vectors. For example figure 1 can be interpreted as follows:

$M^1 y^1 = b^1$ means the restrictions induced by the production technique of a

firm. At $h:=1$ one has to determine the plan, i.e. the level of the decision vector y^1 . Until the end of the first period (at $h:=2$), depending on the

outcome of the random variable nature (especially the transformation matrix A^{2q}) and the level of the decision vector y^1 , lots of different goods ($A^{2q} y^1$)

will be offered on a market. For the outcome q the vector of demand is b^{2q} . As the production plan (level of the decision vector y^1) has to be determined at $h:=1$, supply and demand are not necessarily equal ($A^{2q}y^1 \neq b^{2q}$). The columns of matrix M^{2q} define the different kinds of activities to fill the possible gap between supply and demand for the outcome q of the nature at $h:=2$. y^{2q} means the related decision vector.

2. The Principle of Dynamic Planning

In the above defined decision situation it is obviously no use to determine uniquely the level of the decision vector y^2 (i.e. $y^{2q}=y^2_{wq}$) at $h:=1$. On the other hand it is also no use to determine the level of y^1 at $h:=1$ by neglecting future decisions as the sequence of decisions (the policy) must be optimal for the entire number of planning periods.

One approach to handling this problem is to determine future actions depending on the information (development of the nature and previous decisions) available at that point of time they have to be done. At $h:=1$ the optimal decision has to be made with regard to these future eventual decisions. This approach is called dynamic planning and is well known and discussed as "flexible Planung" in the German management literature in recent times. The principle of dynamic planning is defined by (Hax/Laux, 1972, pp. 319-20; Hax, 1970, pp. 131-33).

Def. 2: A principle of planning is called dynamic (flexible) if at the point of time $\hat{h} \geq 1$

- a) eventual decisions will be made for every outcome of the random variable nature at $\hat{h} = 1, \dots, H$ and
- b) the optimal decision at \hat{h} will be determined with regard to the eventual decisions at $\hat{h} + 1, \dots, H$.

Under the assumption that the decision maker is risk neutral, by using the principle of dynamic planning we obtain for the above discussed example the optimization problem

$$y^1 \in T_1^- \left\{ s_1' y^1 + \sum_{q=1}^Q (p_q s_{2q}' y^{2q}(y^1)) \right\}$$

with

$$T_1^- = \left\{ y^1 \mid y^1 \in T_1, \exists y^{2q} \geq 0 \text{ with } M^{2q} y^{2q} = b^{2q} - A^{2q} y^1 \forall q \right\}$$

and the Q eventual-decision functions depending on y^1

$$s_{2q}' y^{2q}(y^1) = s_{2q}' y^{2q} \rightarrow \text{opt!}$$

$$M^{2q} y^{2q} = b^{2q} - A^{2q} y^1$$

$$y^{2q} \geq 0$$

The optimal production plan at $h:=1$ depends twice on the decision at $h:=2$. First, only such vectors $y^1 \in T_1$ are allowed to be chosen that are associated with feasible solutions $y^{2q} \in T^{2q} \forall q$ at $h:=2$; and second, the optimal plan y^1 has to be determined in consideration of the eventual decision functions at $h:=2$. (We assume $T_1 \neq \emptyset$ and restricted. For $T_1 = \emptyset$ there is no feasible solution. The assumption T_1 restricted can easily be fulfilled by defining an upper bound for the sum of variables in y^1).

By using the principle of dynamic planning the optimal decisions (actions) at $h:=2$ are determined equivalent to Bellman's principle of optimality in dynamic programming (Bellman/Dreyfus, 1971).

Def. 3: An optimal policy has the characteristic that, independent of the origin state of system and the first decision, the remaining decisions have to be an optimal policy for the state that results from the first decision.

3. Characteristics of Dynamic Planning Problems

If $M^{hq} = M^h \Psi_q$ at $h = 1, 2, \dots, H$, we obtain for our example

$$T_{2q} = \left\{ y^{2q} \mid M^2 y^{2q} = b^{2q} - A^{2q} y^1, y^{2q} \geq 0 \right\}$$

which defines a typical two-stage programming problem under risk with discrete distribution function (Werner, 1973, p. 50). But the model of two-stage programming under risk is not restricted to discrete random variables. It is also applicable to continuous random variables and was discussed by Dantzig (1955, pp. 204-5) for $H > 2$.

While Def. 2 defines a general principle of planning for decision situations given by Def. 1, the model of H-stage programming under risk was developed only for stochastic linear programming problems in the above defined situation. Of course it was also applied to problems with nonlinear objective functions in recent times (Sachan, 1970, pp. 211-232), but the assumptions of linear restrictions and deterministic matrices for the decision vectors Ψ_h (only the right-hand-sides and/or the transformation matrices are allowed to be stochastic) are still necessary conditions for its application. So we are able to establish the following characteristics for dynamic planning problems analogous to the H-stage programming model if the decision matrices are deterministic at the H points of time.

Under the assumption that the set of feasible decisions (the decision space) is not empty for any outcome of the random variable at H and every feasible decision at H-1 (i.e. in the example $T_1^- = T_1$) by using Bellman's principle of optimality Dantzig (Dantzig, 1966, pp. 578-9) has shown.

Lemma 1: An H-stage convex optimization problem can be transformed to an equivalent (H-1)-stage optimization problem by

neglecting the restrictions of stage H.

If, for instance, $M^{2q} = M^2 = [E \mid -E]$ in the above discussed example (E means an identity matrix), there is for every vector $[b^{2q} - A^{2q}y^1]$ a solution y^{2q} . The functions $s_{2q}^{2q}(y^1)$ can be determined by using the theory of linear programming. Then the restrictions at $h=2$ can be neglected (Werner, 1973, pp. 113-128). It is $T_1^- = T_1$. By applying lemma 1 the number of restrictions are considerably reduced.

To be able to reduce an H-stage to a (H-1)-stage optimization problem, it is necessary that the set of feasible decisions is not empty for any outcome of the random variable at the point of time H and every feasible decision at H-1. If $\hat{k}_h \in R^{G_h}$ defines the consequences of the outcome of the random variable at h and the decision at h-1 that has to be compensated by the decision vector y^h and M^h the $(G_h \times L_h)$ decision matrix then Kall (Kall, 1966, pp. 246-272) has shown

Lemma 2: There is a $y^h \geq 0$ with $M^h y^h = \hat{k}_h$ for every $\hat{k}_h \in R^{G_h}$ if $G_h < L_h$, the first G_h columns are - if necessary after permutation - linear independent and there is a nonnegative linear combination of the columns $G_h + 1, \dots, L_h$ that can be represented as a negative linear combination of the first G_h columns.

If the matrices M^h suffices the criteria of lemma 2 $\forall h$, the sets of feasible decisions are not empty at any point of time h. The H-stage optimization problem can be reduced to an equivalent one-stage problem in accordance with lemma 1. Whether a matrix M^h with rank G_h and $G_h < L_h$ suffices the above mentioned criteria can be examined by

a) Form $M^h = \begin{bmatrix} M^{Gh} & M^{Lh} \end{bmatrix}$ with M^{Gh} nonsingular square and the

$$G_h \times (L_h - G_h) \text{ matrix } M^{Lh}$$

b) Optimize

$$M^{Lh} y^{Lh} = M^{Gh} y^{Gh}$$

$$y^{Lh} \geq 0$$

$$e r \geq E y^{Gh}$$

$$r \rightarrow \min!$$

with the variable r , the G_h -component column vector $e = (1, 1, \dots, 1)$

and the $G_h (L_h)$ -component vector of variables $y^{Gh} (y^{Lh})$.

c) If $\rho < 0$ the matrix M^h will suffice the criteria of lemma 2.

If M^h doesn't suffice the criteria of lemma 2 we have to derive additional restrictions for the decision variables at $h-1$ to avoid the decision spaces at h becoming empty for any outcome of the random variable nature and any feasible previous decision.

For $H=2$ to handle this problem, first approaches were developed for special structures of M^h by using the Moore-Penrose (generalized) inverse (see appendix, example 1). These approaches are applicable to $H > 2$ also. If the random variable nature is a discrete one at every point of time, it is much simpler to derive the necessary additional restrictions. Especially for $H=2$ rather simple algorithms can be derived by dualizing the optimization problem (see appendix, example 2) and using the Dantzig/Wolfe decomposition principle (Dantzig/Madansky, 196, pp. 165-176). All these approaches are only applicable to special structured matrices M^h and/or discrete random variables. Also lemma 1 was only proved while equivalent reduced optimization problems were derived solely for some simple structured problems.

Appendix

EXAMPLE 1:

Given

$$M y = f ; f := \text{vector of constants}$$

$$y \geq 0$$

with the Moore-Penrose (Generalized)-Inverse M^* of the matrix M , the set of feasible solutions can be represented by

$$\{y\} = M^* f + [E - M^* M] w \geq 0; w \geq 0.$$

If the random variable nature has a continuous density function, we obtain for our example the optimization problem

$$y_1 \in T_1^- \left\{ s_1 y^1 + \int_{-\infty}^{+\infty} g(\xi) \cdot s_2' y^2(y^1) d\xi \right\}.$$

$g(\xi)$ means the density functions of the random variable $\xi := b^2 - A^2 y^1$.

The set T_1^- of feasible solutions y^1 is defined by

$$T_1^- = T_1 \cap \{M^2 y^2 = b^2 - A^2 y^1; y^2 \geq 0\}.$$

By using the Moore-Penrose-Inverse we obtain

$$\begin{aligned} \{y^2\} &= M^{2*} [b^2 - A^2 y^1] + [E - M^{2*} M^2] w \geq 0 \\ M^{2*} [b^2 - A^2 y^1] + [E - M^{2*} M^2] w &\geq 0, \end{aligned}$$

the additional restrictions for y^1 to reduce T_1 to T_1^- . If for example M^{2*} is the left inverse of M^2 they will be reduced to

$$M^{2*} b^2 \geq M^{2*} A^2 y^1$$

T_1^- is defined by

$$T_1^- = \{y^1 \mid M^1 y^1 = b^1; M^{2*} A^2 y^1 \leq M^{2*} b^2; y^1 \geq 0\}.$$

EXAMPLE 2:

For our example with a discrete random variable nature the optimization problem can be written as

$$\begin{array}{ll}
 s_1' y^1 + P_1 \cdot s_{21}' y^{21} + \dots + P_Q \cdot s_{2Q}' y^{2Q} & \rightarrow \text{opt!} \\
 M^1 y^1 & = b^1 \\
 A^{21} y^1 + M^{21} y^{21} & = b^{21} \\
 \vdots & \vdots \\
 A^{2Q} y^1 + \dots + M^{2Q} y^{2Q} & = b^{2Q} \\
 y^1, & y^{21}, \dots, y^{2Q} \geq 0
 \end{array}$$

The dual of this linear program has a blockangular structure and can easily be solved by using modifications of the Dantzig/Wolfe decomposition principle.

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